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Diffusion from a vertical wall into an accelerating falling liquid film

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Abstract—An exact similarity solution is provided for the mass transfer from a vertical surface to a falling film. The velocity field of the accelerating gravity-driven film is exactly represented by a Falkner–Skan type stream function f, from which the local Nusselt number is obtained as

$$Nu_x = \left(\frac{3}{4}Re_x\right)^{1/2} \cdot \left\{\int_0^\infty \exp\left[-Sc\int_0^t f(z)\,dz\right]dt\right\}^{-1}.$$

Accurate numerical values are given for Schmidt numbers from 0.001 to 1000, while simple asymptotic formulas are provided for the extreme Schmidt numbers.

INTRODUCTION

THE FLOW of liquids in thin films is a well-known phenomenon in everyday life as well as in numerous technological applications. Of particular importance, for instance in chemical engineering, is the mass or heat transfer associated with many falling film concepts.

The solid–liquid mass transfer between a plane surface and a falling film has been dealt with theoretically and experimentally in several papers. Kramers and Kreyger [1] and Oliver and Atherinos [2] studied the dissolution of a soluble wall and the subsequent penetration of the solute into the liquid film. Iribarne *et al.* [3] and Wragg *et al.* [4] considered the diffusion-controlled electrolytic mass transfer between a wall and a falling film, while Alekseenko *et al.* [5] derived the wall shear stress from measured data for the diffusion-controlled electrolytic current.

The theoretical considerations in refs. [1–5] have naturally been based on some simplifying assumptions. Two of the basic simplifications are that: (a) the flow field is fully developed in the sense that the streamwise derivatives of the velocity components are negligible; and (b) the gradients of concentration exist only near the wall where the shear stresses in the liquid do not differ appreciably from the value τ_w at the wall. With these assumptions, the film thickness h is taken as being a constant, and the velocity components are approximated as $u = y\tau_w/\mu$ and $v = 0$ in the actual near-wall region. Using the above simplifications Kramers and Kreyger [1] derived an explicit analytic expression for the surface mass flux, which has subsequently been employed and modified by others [2–5]. However, their solution is neither applicable when the concentration boundary layer penetrates deeply into the liquid phase nor to developing film flows.

In this paper we consider the mass transfer between a gravity-driven accelerating film and a vertical surface. The intention of the analysis is to demonstrate that the similarity solution of Fage and Falkner [6] for the heat transfer behaviour of a laminar momentum boundary layer on an isothermal wedge, also solves the concentration boundary layer in developing liquid films. The similarity solution which in this way is made available, also applies if the concentration profile spreads into parts of the film in which the shear stresses are significantly smaller than that at the wall.

PHYSICAL MODEL AND BASIC EQUATIONS

We consider an accelerating film of a Newtonian liquid flowing down along a smooth vertical surface, as depicted in Fig. 1. Uniform flow enters the system at $x = 0$, the total volumetric liquid feed being denoted by Q. The flow is laminar and the free surface of the film is wavefree. Provided that the thickness $\delta(x)$ of the viscous boundary layer which develops along the wall is smaller than the local film thickness $h(x)$, a quasi-one-dimensional inviscid flow exists between the momentum boundary layer and the free streamline bordering the constant-pressure atmosphere.

The two flow regimes, which are qualitatively different, can be treated separately as in aerodynamic analyses. For the inviscid part of the flow the Bernoulli equation yields

$$\frac{1}{2}U^2 - gx = \text{constant} \tag{1}$$

while the viscous flow within the boundary layer is

NOMENCLATURE

c concentration [kg m^{-3}]
D molecular diffusion coefficient [m^2 s^{-1}]
f dimensionless stream function, equation (7a)
g gravitational acceleration [m s^{-2}]
h local film thickness [m]
I integral, equation (12b)
$\bar{k}$ average mass transfer coefficient, equation (15) [m s^{-1}]
L streamwise length for averaging k
m exponent
Nu_x local Nusselt number, equation (13)
Q volumetric flow rate [m^2 s^{-1}]
Re_x local Reynolds number, $xU(x)/\nu$
Sc Schmidt number, ν/D
U free stream velocity [m s^{-1}]
u streamwise velocity component [m s^{-1}]
v cross-stream velocity component [m s^{-1}]
x streamwise coordinate [m]
y cross-stream coordinate [m].

Greek symbols
δ boundary layer thickness [m]
δ_1 boundary layer displacement thickness, $\int_0^\infty (1-f')\,d\eta$
η dimensionless coordinate, equation (7b)
θ dimensionless concentration, equation (7c)
μ molecular viscosity [kg m^{-1} s^{-1}]
ν kinematic viscosity, μ/ρ [m^2 s^{-1}]
τ shear stress [kg m^{-1} s^{-2}]
ψ stream function [m^2 s^{-1}].

Subscripts
c concentration
o condition outside the boundary layers
w condition at the wall.

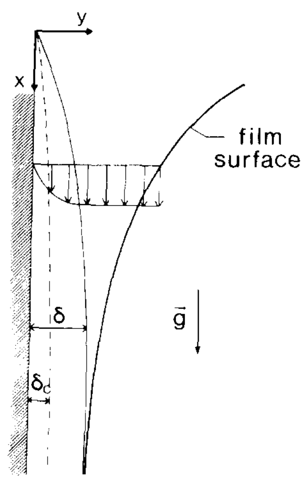

FIG. 1. Coordinate system and notations.

governed by the reduced Navier–Stokes equation for incompressible flow

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g+\nu\frac{\partial^2 u}{\partial y^2} \quad (2)$$

with the boundary conditions

$$u(x,0)=0 \quad (3a)$$

$$v(x,0)=0 \quad (3b)$$

$$u(x,y)\to U(x) \text{ as } y\to\delta. \quad (3c)$$

Here, we have implicitly assumed that the diffusion rate at the wall results in a normal velocity component which is negligible.

In accordance with classical boundary layer arguments, the streamwise diffusion of momentum has been neglected in equation (2). Using the same sort of arguments for the concentration conservation equation, the governing equation reduces to

$$u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}=D\frac{\partial^2 c}{\partial y^2} \quad (4)$$

where $c(x,y)$ denotes the concentration distribution and D is the diffusion coefficient of the diffusing species in the liquid. The incoming flow is assumed to have a uniform concentration c_o of the diffusing species, while the concentration c_w is maintained at the wall. Thus, the relevant boundary conditions for the concentration equation (4) become

$$c(x,0)=c_w \quad (5a)$$

$$c(x,y)\to c_o \text{ as } y\to\delta_c \quad (5b)$$

where δ_c denotes the thickness of the concentration boundary layer. Considering the dissolution of a soluble wall [1, 2], c_w represents the saturation concentration, or solubility, of the wall substance in the liquid. In the electrochemical case [3–5], however, the concentration of reacting ions at the wall (i.e. the electrode) is assumed negligible, i.e. $c_w \approx 0$.

Assuming zero velocity (and infinite film thickness) at the entrance $x = 0$, the simple solution

$$U(x)=\sqrt{(2gx)} \quad (6)$$

is readily derived from the Bernoulli equation (1). Andersson and Ytrehus [7] recently recognized the formal equivalence of the inviscid velocity distribution

(6) with the freestream variation $U(x) \sim x^m$ considered by Falkner and Skan [8] for laminar boundary layer flow along wedge-shaped bodies. Following ref. [7], we introduce the similarity variables f and η

$$\psi = f\sqrt{\left(\frac{2}{m+1}U\nu x\right)} \tag{7a}$$

$$\eta = y\sqrt{\left(\frac{m+1}{2}U/\nu x\right)} \tag{7b}$$

where ψ is the physical stream function. In accordance with equation (6), the power m is set equal to 1/2 in the present context. Anticipating that similarity can be achieved also for the concentration distribution, we assume

$$c(x,y) = c_o + (c_w - c_o)\,\theta(\eta) \tag{7c}$$

where $\theta(\eta)$ is a dimensionless concentration.

The momentum and concentration equations can now be rewritten in terms of the dimensionless dependent variables f and θ. The resulting ordinary differential equations and boundary conditions are

$$f''' + ff'' + \frac{2}{3}(1 - f'^2) = 0 \tag{8}$$

$$\theta'' + Sc\cdot f\theta' = 0 \tag{9}$$

$$f(0) = f'(0) = 0 \tag{10a}$$

$$f'(\eta) \to 1 \text{ as } \eta \to \infty \tag{10b}$$

$$\theta(0) = 1 \tag{11a}$$

$$\theta(\eta) \to 0 \text{ as } \eta \to \infty. \tag{11b}$$

Evidently, the Schmidt number $Sc \equiv \nu/D$ is the only explicit parameter in the transformed problem.

It should be emphasized that the momentum equation (8) is uncoupled from the mass transfer equation (9), while the concentration field θ is on the other hand coupled to the velocity field through the stream function f in equation (9).

THE EXACT SOLUTION

The momentum boundary layer problem, i.e. equation (8) subject to the boundary conditions (10), has been solved recently by Andersson and Ytrehus [7]. The numerical solution is exact in the sense that the partial differential equation (2) in the primitive variables u and v is exactly represented by the ordinary differential equation (8) in the variable f, whose numerical solution can be obtained to an arbitrary degree of accuracy.

By taking advantage of the formal equivalence of the concentration boundary layer problem (9, 11) with the thermal boundary layer problem along a semi-infinite wedge, the solution of the thermal problem may be carried directly over to the mass transfer problem in accelerating film flow. With the solution of the velocity field known, the solution of equation (9) subject to the boundary conditions (11) can be written as

$$\theta(\eta) = 1 - \frac{I(\eta;Sc)}{I(\infty;Sc)} \tag{12a}$$

where $I(\eta;Sc)$ represents the integral term

$$I(\eta;Sc) \equiv \int_0^\eta \exp\left[-Sc\int_0^t f(z)\,dz\right]dt. \tag{12b}$$

The integral relation (12) was originally derived by Pohlhausen [9] for the flat plate thermal boundary layer, which represents the special case $m = 0$ of the Falkner–Skan wedge-flow similarity solutions. However, solution (12) may represent any flow along an isothermal wall in which the external velocity $U(x)$ is proportional to x^m [6]. The dependence on the wedge angle is thus implicitly taken into account through the stream function f.

Of particular interest in technological applications is the heat transfer between the wedge and the fluid, which in the present context is analogous to the mass transfer between the vertical wall and the liquid film. The local mass transfer is conveniently expressed in dimensionless form as a local Nusselt number

$$Nu_x \equiv -\frac{x}{c_w - c_o}\left(\frac{\partial c}{\partial y}\right)_{y=0} \tag{13}$$

which is obtained from solution (12) as

$$Nu_x = -\theta'(0)\cdot\sqrt{\left(\frac{3}{4}Re_x\right)} \tag{14a}$$

$$\theta'(0) = -1/I(\infty;Sc) \tag{14b}$$

where $Re = xU(x)/\nu$ is the local Reynolds number and $\theta'(0)$ is the dimensionless concentration gradient evaluated at the wall. For the particular parameter value $m = 1/2$ the heat transfer relation for wedge surfaces of Fage and Falkner [6] as cited in ref. [10, p. 246] is identical in form to the present mass transfer relation (14).

It may sometimes be convenient to express the mass flow rate between the solid surface and the liquid film in terms of a (dimensional) average mass transfer coefficient

$$\bar{k} \equiv -\frac{1}{L}\int_0^L \frac{D}{c_w - c_o}\frac{\partial c}{\partial y}\bigg|_{y=0} dx = \frac{D}{L}\int_0^L \frac{Nu_x}{x}dx. \tag{15}$$

The average surface coefficient calculated from equation (14) becomes

$$\bar{k} = (32g\nu^2/9L)^{1/4}/[Sc\cdot I(\infty;Sc)]. \tag{16}$$

Finally, it should be emphasized that the preceding analysis for mass transfer into a falling film may also be applied to the corresponding thermal-energy problem, i.e. the heat transfer through the film–wall interface. However, in order to achieve the similarity

Table 1. Computed values of the integral $I(\infty; Sc)$ and local Nusselt number. The required data from the exact solution of the velocity field [7] are $\delta_1 = 0.74017$ and $f''_w = 1.03890$

Sc	$I(\infty; Sc)$	$Nu_x \cdot Re_x^{-1/2}$
0.001	4.03663(+1)	2.14542(−2)
0.003	2.36101(+1)	3.66804(−2)
0.006	1.69029(+1)	5.12354(−2)
0.01	1.32508(+1)	6.53566(−2)
0.03	7.93772	1.09103(−1)
0.06	5.80325	1.49231(−1)
0.1	4.63550	1.86825(−1)
0.3	2.91726	2.96863(−1)
0.6	2.21012	3.91845(−1)
1	1.81339	4.77572(−1)
3	1.20403	7.19274(−1)
6	9.37848(−1)	9.23417(−1)
10	7.82520(−1)	1.10671
30	5.33468(−1)	1.62339
60	4.20252(−1)	2.06073
100	3.52900(−1)	2.45402
300	2.42966(−1)	3.56440
600	1.92224(−1)	4.50530
1000	1.61819(−1)	5.35183

of the transport equations for heat and mass, it is required that heat generation due to viscous dissipation is negligible in the thermal energy equation. Then, if the Schmidt number is replaced with the Prandtl number, the solution for $\theta(\eta)$ in equation (12) is the dimensionless temperature distribution.

RESULTS AND DISCUSSIONS

The final results of the present analysis, i.e. equations (14) and (16), are exact provided that the integral $I(\infty; Sc)$ is computed from equation (12b) using an exact representation of the dimensionless stream function f as obtained from the momentum equation (8). Numerical values of the wall gradient $\theta'(0)$, which is the reciprocal of the integral $I(\infty; Sc)$, has been tabulated by Evans in his textbook [11] for a wide range of the parameters m and Sc. These extensive tables contain results for $m = 3/7$ and $2/3$, while the particular parameter value $m = 1/2$ is unfortunately not included. An accurate integration algorithm for the evaluation of integral (12b) was therefore constructed, in accordance with the procedure outlined by Evans [12]. With the non-dimensional displacement thickness $\delta_1 \equiv \int_0^\infty (1 - f')\,d\eta$ and wall shear stress $f''_w \equiv f''(0)$ known from the similarity solution [7] of the velocity equation, $I(\infty; Sc)$ and Nu_x were evaluated for Schmidt numbers ranging from 0.001 to 1000. The numerical values are given in Table 1. Corresponding calculations for $m = 3/7$ agreed in the fifth significant digit with the values obtained by Evans [11], thus indicating the accuracy of the results.

While Table 1 gives values of the reciprocal of the wall gradient, similarity profiles of the dimensionless concentration θ is displayed in Fig. 2, from which the Schmidt number effect is evident. For $Sc \ll 1$ the concentration boundary layer extends far into the inviscid flow, while for high Schmidt numbers the concentration layer is significantly thinner than the viscous boundary layer. This qualitative difference between high and low Schmidt number results is also exhibited in Fig. 3, where the computed values of the local Nusselt number in Table 1 have been plotted vs the Schmidt number.

In the limiting case $Sc \to 0$ the velocity boundary layer becomes so thin compared to the concentration boundary layer, that it can be neglected in solving the concentration equation. The resulting asymptotic solution

$$Nu_x = \left(\frac{m+1}{\pi} Re_x Sc\right)^{1/2} \quad (17)$$

is a well-known result for the heat transfer rate in Falkner–Skan wedge flow boundary layers; see e.g. ref. [11].

In the high Schmidt number (low diffusivity) limit, the concentration gradients are contained within the innermost part of the velocity boundary layer. For this Schmidt number regime, Evans [11] derived an asymptotic expansion for the reciprocal wall gradient in inverse powers of Sc. Keeping the highest order term only, the local Nusselt number becomes

$$Nu_x = \frac{3}{\Gamma\left(\frac{1}{3}\right)} \left(\frac{m+1}{2} Re_x\right)^{1/2} \left(\frac{Sc f''_w}{3!}\right)^{1/3} \quad (18)$$

where the required value of the gamma function is $\Gamma(1/3) = 2.67894$. Thus, for the particular parameter value $m = 1/2$ we obtain

$$Nu_x = 0.540542 Re_x^{1/2} Sc^{1/3} \quad (19)$$

in the limit $Sc \to \infty$.

The characteristic asymptotes (17) and (19) are shown as lines in Fig. 3. It is evident that the asymptotic formulas closely approximate the exact solution (symbols) in the extreme Schmidt number regimes, while more accurate expressions are required for Schmidt numbers of the order unity.

It may be worthwhile to compare the expression for the surface coefficient derived by Kramers and Kreyger [1]

$$\bar{k} = 0.914[Qg^2\nu^4/L^3]^{1/9} Sc^{-2/3} \quad (20)$$

with the present solution. In the limit $Sc \to \infty$ equation (16) becomes

$$\bar{k} = 0.857[g\nu^2/L]^{1/4} Sc^{-2/3} \quad (21)$$

while $\bar{k} \sim Sc^{-1/2}$ in the low Schmidt number range. Being analogous to the corresponding heat transfer result, equation (20) is sometimes known as the Lévêque solution.

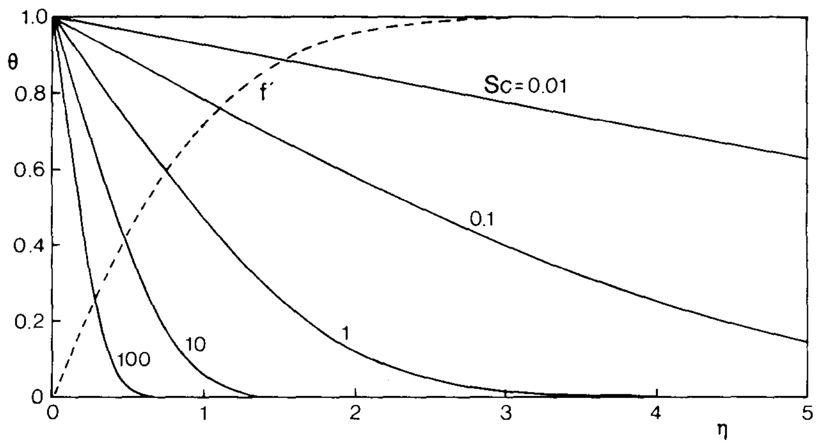

FIG. 2. Dimensionless concentration profiles $\theta(\eta)$ (solid lines) and velocity profile $f'(\eta)$ (broken line).

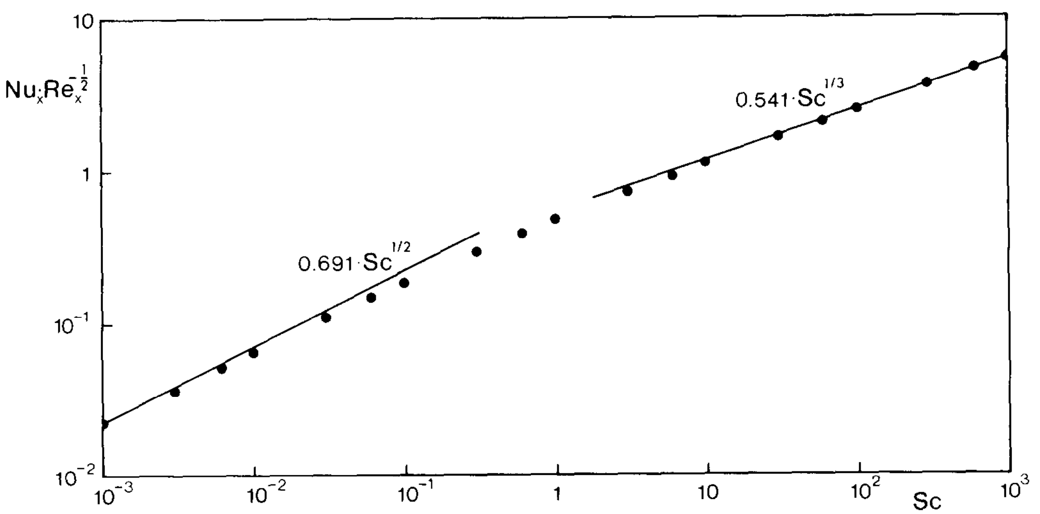

FIG. 3. The dependence of local Nusselt number on Schmidt number. The symbols denote exact results and the lines represent the extreme asymptotes (17) and (19).

It is observed that the asymptotic formula (21) exhibits the same Schmidt number dependence as equation (20). The analysis of Kramers and Kreyger was based on the assumption of a very thin concentration layer, i.e. that the solute penetration is restricted to a region with approximately linear velocity variation. Examination of Evan's [11] asymptotic expansion, reveals that the high Schmidt number asymptotes (18), (19) and (21) correspond to the neglection of all but the leading term $f'(\eta) = \eta \cdot f''_w$ in the power series for f'.

Even though both equations (20) and (21) are based on a linear variation of the velocity through the concentration layer, the flow fields are still quite different. While Kramers and Kreyger assumed that $v = 0$ and $\partial u/\partial x = 0$, none of these assumptions were imposed in the preceding analysis.

Finally, it is important to be aware of a limitation of the present results. For solutions (12) and (14) to be valid, the solution domain must be sufficiently restricted in the downstream direction for the boundary conditions (3c) and (5b) to be satisfied in some asymptotic sense within the total thickness of the film. It is thus required that δ and δ_c should be less than the local film thickness $h(x)$.

Andersson and Ytrehus [7] showed that the region in which an inviscid flow exists between the velocity boundary layer and the surrounding atmosphere, extends a distance x_o in the downstream direction, x_o being given as

$$x_o = 0.1972(3\nu Q/g)^{1/3}\, Q/\nu. \quad (22)$$

Here, $(3\nu Q/g)^{1/3}$ is recognized as the thickness of the fully developed laminar film and Q/ν is the film Reynolds number.

For $Sc \gg 1$ the concentration layer is much thinner than the momentum boundary layer, i.e. $\delta_c \ll \delta$, and both conditions (3c) and (5b) are satisfied within the streamwise range $0 < x < x_o$. For $Sc \ll 1$, however, δ_c becomes significantly greater than δ, and the concentration gradients will therefore reach the film surface upstream of x_o. In order to ensure the validity of equation (5b), the streamwise extension of the solution domain should therefore be further restricted for Schmidt numbers below unity.

CONCLUDING REMARKS

The primary purpose of this paper is to present a new similarity solution for the diffusion of mass (or heat) from a vertical wall to a gravity-driven liquid film. The main contribution may be summarized as follows.

(1) An exact similarity solution for mass transfer is provided, in which both velocity components have been exactly represented by the solution of the Falkner–Skan equation for the particular parameter value $m = 1/2$.

(2) Asymptotic formulas for the local Nusselt number in the extreme Schmidt number regimes are derived. Comparison with the exact solution reveals that the Schmidt number dependence decreases from $Sc^{1/2}$ to $Sc^{1/3}$ over the whole range.

(3) Accurate numerical values of the local Nusselt number are provided, covering the range of Schmidt numbers from 0.001 to 1000.

(4) The exact similarity solution constitutes a reference against which any approximate solution may be checked.

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DIFFUSION A PARTIR D'UNE PAROI VERTICALE DANS UN FILM LIQUIDE TOMBANT AVEC ACCELERATION

Résumé—Une solution exacte de similitude est donnée pour le transfert massique à partir d'une surface verticale vers un film tombant. Le champ de vitesse du film accéléré soumis à la pesanteur est exactement représenté par une fonction de courant f de type Falkner–Skan d'où est déduit le nombre de Nusselt local:

$$Nu_x = \left(\frac{3}{4}Re_x\right)^{1/2} \cdot \left\{\int_0^\infty \exp\left[-Sc\int_0^t f(z)\,dz\right]dt\right\}^{-1}.$$

Des valeurs numériques précises sont données pour des nombres de Schmidt entre 0,001 et 1000, et des formules asymptotiques simples sont données pour les nombres de Schmidt extrêmes.

DIFFUSION VON EINER SENKRECHTEN WAND IN EINEN BESCHLEUNIGTEN RIESELFILM

Zusammenfassung—Eine exakte Ähnlichkeitsbeziehung für den Stofftransport von einer senkrechten Wand in einen Rieselfilm wird vorgelegt. Das Geschwindigkeitsfeld des durch Schwerkraft beschleunigten Rieselfilms wird exakt durch eine Stromfunktion f vom Falkner–Skan-Typ wiedergegeben, aus der die örtliche Nusselt-Zahl wie folgt berechnet wird:

$$Nu_x = \left(\frac{3}{4}Re_x\right)^{1/2} \cdot \left\{\int_0^\infty \exp\left[-Sc\int_0^t f(z)\,dz\right]dt\right\}^{-1}.$$

Exakte Zahlenwerte werden für Schmidt-Zahlen von 0,001 bis 1000 angegeben, während einfache Näherungsgleichungen für extreme Schmidt-Zahlen vorgelegt werden.

ДИФФУЗИЯ ОТ ВЕРТИКАЛЬНОЙ СТЕНКИ В ПЛЕНКУ ЖИДКОСТИ, СТЕКАЮЩУЮ С УСКОРЕНИЕМ

Аннотация—Приведено автомодельное решение для массопереноса от вертикальной поверхности в стекающую пленку. Поле скорости пленки, увлекаемой ускорением силы тяжести, представлено функцией тока типа Фолкнера-Скэна f, из которой получается локальное число Нуссельта в виде

$$Nu_x = (\tfrac{3}{4}Re_x)^{1/2} \cdot \left\{\int_0^\infty \exp\left[-Sc\int_0^t f(z)\,\mathrm{d}z\right]\mathrm{d}t\right\}^{-1}.$$

Численные значения даны для чисел Шмидта от 0,001 до 1000, в то время как простые асимптотические зависимости представлены для экстремальных чисел Шмидта.